Kinematics and uncertainty relations of a quantum test particle in a curved space-time *

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Abstract

A possible model for quantum kinematics of a test particle in a curved space-time is proposed. Every reasonable neighbourhood V_e of a curved space-time can be equipped with a nonassociative binary operation called the geodesic multiplication of space-time points. In the case of the Minkowski space-time, left and right translations of the geodesic multiplication coincide and amount to a rigid shift of the space-time $x \to x + a$. In a curved space-time infinitesimal geodesic right translations can be used to define the (geodesic) momentum operators. The commutation relations of position and momentum operators are taken as the quantum kinematic algebra. As an example, detailed calculations are performed for the space-time of a weak plane gravitational wave. The uncertainty relations following from the commutation rules are derived and their physical meaning is discussed.

1 Introduction

The Poincaré group − the symmetry group of the flat space-time M − and its representations are basic constitutive elements for relativistic theories. A generic curved space-time V doesn't allow symmetry groups and the Poincaré group looses its central role. The Lorentz group can be considered as the symmetry group of flat tangent spaces, but the status of the Poincaré translations is unclear.

The Poincaré translations form an Abelian group and describe rigid shifts along straight lines of a flat space-time, $x \to x + a$, $a = const$. Straight lines are geodesic lines of the Minkowski space. In a curved space-time analogous geodesic translations can be introduced in a (finite) neighbourhood V_e of $e \in V$ using the concept of geodesic multiplication of points $x, y \in V_e$. The neighbourhood V_e together with the binary operation of geodesic multiplication constitutes an algebraic system called local geodesic loop. In general, it is noncommutative and nonassociative [1, 2, 3]. As a result we obtain a novel generalization of Poincaré shifts to the case of a curved space-time (Sec. 2).

In the present paper we investigate some prospects of using local geodesic loops for constructing a quantum kinematics in the background of a curved space-time (Sec. 3). Let us introduce an action of the position operators $\mathbf{x}^i$ (on scalar valued functions) as multiplication with the Riemann normal coordinates x^i. We propose to define (geodesic) momentum operators $\mathbf{p}_k$ via infinitesimal right geodesic translations by $\mathbf{p}_k(x) = -i\hbar R^s_k(x)\partial_s$. The corresponding commutation relations are taken as the quantum kinematic algebra [4]:

$$[\mathbf{x}^i, \mathbf{x}^k] = 0, \quad [\mathbf{x}^i, \mathbf{p}_k] = i\hbar R^i_k(x), \quad [\mathbf{p}_j, \mathbf{p}_k] = -i\hbar \rho^n_{jk}(x)\, \mathbf{p}_n.$$

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The uncertainty relations which follow from the modified $[\mathbf{x}^i, \mathbf{p}_k]$ commutator can put restrictions on minimal values of coordinates and momenta [5, 6, 7].

As an example, detailed calculations are performed in the case of a weak plane gravitational wave background (Sec. 4).

There seems to be also another possibility of using the concept of local geodesic loops for constructing quantum kinematics in the background of a curved space-time. In the flat space-time quantum field theory, one-particle states are introduced as representations of the Poincaré group and their momentum is identified as eigenvalues of the Poincaré translation operators $\mathbf{P}_\mu$. We could mimic it by defining one-particle states in a curved background via representations of geodesic loops. However, since the representation theory of general nonassociative structures is still essentially lacking, we cannot hope a quick progress along these lines (Sec. 5).

2 Geodesic loops and geodesic translations

Let us consider a manifold V with a symmetric (torsionless) affine connection $\Gamma^\lambda_{\mu\nu}(x) = \Gamma^\lambda_{\nu\mu}(x)$. Let $x, y \in M_e$ be two points of a neighbourhood V_e of $e \in V$ such that geodesic archs between each two points are uniquely determined. Geodesic multiplication in respect of the unit element e is defined by the following formula [1, 2]

$$x \cdot y \equiv L_x y \equiv R_y x = \left(\exp_y \circ \tau^e_y \circ \exp_e^{-1}\right) x \,. \tag{1}$$

Here $\exp_e X$ denotes exponential mapping $X \to x$, $X \in T_eV$, $x \in V$, and $\tau^e_y : T_eV \to T_yV$ is the parallel transport mapping of tangent vectors from T_eV into T_yV along the unique local geodesic arch joining the points e and y. By L_x and R_y we have defined the left (L) and the right (R) translation operators in analogy with the case of groups.

From the definition of the geodesic multiplication (1) it follows that in the case of the Minkowski space-time with orthonormal coordinates x, the right and left geodesic translations coincide, $R^{flat}_a = L^{flat}_a$ and the geodesic multiplication $x \to x \cdot a = a \cdot x$ describes a rigid shift of the space-time, $x \to x + a$.

Let us introduce the following infinitesimal right translation matrix:

$$(x \cdot y)^\mu = x^\mu + R^\mu_\nu(x) y^\nu + \ldots , \qquad R^\mu_\nu(x) \equiv \frac{\partial (x \cdot y)^\mu}{\partial y^\nu}\Big|_{y=e} . \tag{2}$$

Matrix $R^\mu_\nu(x)$ can be used to introduce a local frame field [8, 9]

$$R_\nu(x) \equiv R^\mu_\nu(x) \partial_\mu \,. \tag{3}$$

From Eq. (2) it follows that in the unit element e we have $R^\mu_\nu(e) = \delta^\mu_\nu$. The commutator of vector fields $R_\nu(x)$ define the structure functions $\rho^\sigma_{\mu\nu}(x)$:

$$[R_\mu(x), R_\nu(x)] = \rho^\sigma_{\mu\nu}(x) R_\sigma(x) \,. \tag{4}$$

Let us specify the coordinates $x \in V_e$ to be the Riemann normal coordinates, i.e. the equations of geodesics emerging from e are

$$x^i(t) = X^i t, \qquad X^i \in T_eV \,. \tag{5}$$

Now the equations of exponential mapping and parallel transport which determine the geodesic multiplication (1) can be integrated in the neighbourhood of e as power series in x [10, 3] and the following expansion for structure functions $\rho^k_{ij}(x)$ can be calculated:

$$\rho^k_{ij}(x) = 2R^k{}_{n[ij]}(e) x^n + \ldots \,. \tag{6}$$

Here $R^k_{nij}(e)$ denote the components of the Riemann curvature tensor at the origin of coordinates e. Note that from the algebraic point of view, they are the main part of the associator of the geodesic multiplication [10]:

$$\Big(\big(x(yz)\big)^{-1}_L \big((xy)z\big) \Big)^m = R^m{}_{nrs}(e) x^n y^r z^s + \ldots \,. \tag{7}$$

In this sense the emergence of structure functions instead of structure constants in the commutator (4) is caused by the nonassociativity of geodesic multiplication.

3 Kinematics of a quantum test particle

Let us introduce an action of the position operators $\mathbf{x}^i$ (on scalar valued functions) as multiplication with the Riemann normal coordinates x^i. Then we have $[\mathbf{x}^i, \mathbf{x}^k] = 0$. We propose to define (geodesic) momentum operators $\mathbf{p}_i$ via infinitesimal right geodesic translations (2) by $\mathbf{p}_k = -i\hbar R^s_k(x)\partial_s$. Then,

$$[\mathbf{x}^i, \mathbf{p}_k] = [\mathbf{x}^i, -i\hbar R^s_k(x)\partial_s] = i\hbar R^i_k(x). \tag{8}$$

The full (geodesic) kinematic algebra of a quantum test particle in a curved space-time now reads [4]

$$[\mathbf{x}^i, \mathbf{x}^k] = 0, \quad [\mathbf{x}^i, \mathbf{p}_k] = i\hbar R^i_k(x), \quad [\mathbf{p}_j, \mathbf{p}_k] = -i\hbar \rho^n_{jk}(x)\, \mathbf{p}_n. \tag{9}$$

Note that the modification of commutation relations does not introduce any new dimensionful constants.

In a torsionless space-time, an expansion of $R^i_k(x)$ in the Riemann normal coordinates ($\Gamma^m_{nr}(e) = 0$) can be found by using Eq. (2) and the geodesic multiplication formula of Akivis [10]

$$(x\alpha)^m = x^m + \alpha^m - \frac{1}{2}\Gamma^m_{nr,s}(e) x^n x^r \alpha^s - \frac{1}{2}\Gamma^m_{n(r,s)}(e) x^n \alpha^r \alpha^s + \dots . \tag{10}$$

We get

$$[\mathbf{x}^i, \mathbf{p}_k] = i\hbar \left(\delta^i_k - \frac{1}{3} J^i{}_{kmn}(e) x^m x^n + O(x^3) \right), \tag{11}$$

where $J^i{}_{kmn}$ denotes the Jacobi curvature tensor,

$$J^i{}_{kmn} = \frac{1}{2}(R^i{}_{mkn} + R^i{}_{nkm}). \tag{12}$$

Using Eq. (6) the expression for $[\mathbf{p}_i, \mathbf{p}_j]$ in the Riemann normal coordinates reads

$$[\mathbf{p}_i, \mathbf{p}_j] = -2i\hbar \left(R^k{}_{n[ij]}(e) x^n + O(x^2) \right) \mathbf{p}_k \,. \tag{13}$$

Approximate expressions (11), (13) for the commutators were presented also by Kempf [7], who introduced momentum operators as generators of the change of geodesic coordinates at infinitesimal shift of their origin and used Synge's world function for calculating commutators.

4 An example: quantum kinematics in the background of a weak plane gravitational wave

The metric tensor of the space-time of a weak plane gravitational wave can be given as perturbations around the Minkowski metric [11]:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

$$\eta_{\mu\nu} = diag(-1, +1, +1, +1),$$

In case of a polarized weak plane gravitational wave moving in the direction of x^1 the only non-zero components of $h_{\mu\nu}$ in the TT-gauge are

$$h_{22} = -h_{33} = A \cos\omega(x^0 - x^1). \tag{14}$$

Here $A = const, A \ll 1$, is the wave amplitude, and all subsequent equations hold in the linear approximation in A.

In these coordinates the equation of a geodesic line $x^\mu(t)$ with a tangent vector X^μ at a point e can be easily integrated, yielding [8]

$$x^\mu(t) = X^\mu t + AU^\mu \frac{\sin\omega Ct - \omega Ct}{\omega C^2},$$

where we have taken $e^\mu = 0$ and denoted

$$C = X^0 - X^1,$$
$$U^0 = U^1 = -\frac{1}{2}\left((X^2)^2 - (X^3)^2\right),$$
$$U^2 = -X^2 C \quad , \quad U^3 = X^3 C.$$

The coordinates X^μ are also the Riemann normal coordinates of a point x with TT-coordinates $x^\mu \equiv x^\mu(1)$.

Let us choose the point e to be the unit element of the geodesic loop and let x, y be two points from its neighbourhood. According to (1), for calculating the product of the points x, y, the corresponding equations of geodesics and of parallel transport of the tangent vector X^μ must be integrated. From the expression for the geodesic product the matrix of the right translations and the corresponding structure functions can be determined [8].

According to our proposal, canonical momentum operators can be represented by $\mathbf{p}_i = -iR_i^s\partial_s$. The generalized commutation relations (9) in the background of a weak plane gravitational wave read

$$\begin{aligned}
[\mathbf{x}^i, \mathbf{x}^k] &= 0, \\
[\mathbf{x}^k, \mathbf{p}_i] &= i\hbar\left(\delta_i^k + A\frac{\sin\omega C - \omega C}{\omega C^3}(2U^k\partial_i C - \partial_i U^k C)\right), \\
[\mathbf{p}_i, \mathbf{p}_j] &= \frac{2i\hbar A}{\omega C^2}\sin^2\frac{\omega C}{2}\left(\partial_j U^k \partial_i C - \partial_i U^k \partial_j C\right)\mathbf{p}_k.
\end{aligned}$$

Let us consider more in detail $[\mathbf{x}^k, \mathbf{p}_i]$ commutator. Near the light-cone emerging from the unit element or in the long wavelength approximation, it can be expanded as a series in ωC, giving a polynomial function on its r.h.s.:

$$[\mathbf{x}^k, \mathbf{p}_i] = i\hbar\left(\delta_i^k - a(2U^k\partial_i C - C\partial_i U^k)\right), \qquad (15)$$

where $a \equiv \frac{1}{6}A\omega^2 > 0$. Introducing light-cone coordinates $u = X^0 - X^1$, $v = X^0 + X^1$, $p_u = \frac{1}{2}(p_0 - p_1)$, $p_v = \frac{1}{2}(p_0 + p_1)$ and denoting $X^2 = y$, $X^3 = z$, the nonvanishing commutators read

$$\begin{aligned}
[\mathbf{u}, \mathbf{p}_u] &= i\hbar, & [\mathbf{v}, \mathbf{p}_u] &= 2i\hbar a(\mathbf{y}^2 - \mathbf{z}^2), \\
[\mathbf{y}, \mathbf{p}_u] &= i\hbar a\mathbf{y}\mathbf{u}, & [\mathbf{z}, \mathbf{p}_u] &= -i\hbar a\mathbf{z}\mathbf{u}, \\
[\mathbf{v}, \mathbf{p}_v] &= i\hbar, & & \\
[\mathbf{v}, \mathbf{p}_y] &= -2i\hbar a\mathbf{y}\mathbf{u}, & [\mathbf{y}, \mathbf{p}_y] &= i\hbar(1 - a\mathbf{u}^2), \\
[\mathbf{v}, \mathbf{p}_z] &= 2i\hbar a\mathbf{z}\mathbf{u}, & [\mathbf{z}, \mathbf{p}_z] &= i\hbar(1 + a\mathbf{u}^2).
\end{aligned}$$

From these commutators the following uncertainty relations can be derived:

$$\begin{aligned}
\Delta u \Delta p_u &\geq \frac{\hbar}{2}, \\
\Delta v \Delta p_u &\geq \hbar a\left((\Delta y)^2 - (\Delta z)^2 + <y>^2 - <z>^2\right), \\
\Delta y \Delta p_u &\geq \frac{\hbar a}{2}\left(\Delta y \Delta u + <y><u>\right), \\
\Delta z \Delta p_u &\geq \frac{\hbar a}{2}\left(\Delta z \Delta u + <z><u>\right), \\
\Delta v \Delta p_v &\geq \frac{\hbar}{2}, \\
\Delta v \Delta p_y &\geq \hbar a\left(\Delta y \Delta u + <y><u>\right), \\
\Delta y \Delta p_y &\geq \frac{\hbar}{2}\left(1 - a((\Delta u)^2 + <u>^2)\right), \\
\Delta v \Delta p_z &\geq \hbar a\left(\Delta z \Delta u + <z><u>\right), \\
\Delta z \Delta p_z &\geq \frac{\hbar}{2}\left(1 + a((\Delta u)^2 + <u>^2)\right).
\end{aligned}$$

Consider a state which has $<u>= 0$, $<y>= 0$ or $<z>= 0$. An example of such a state is a test particle moving along with the wave front. The uncertainties for p_u then read

$$\Delta p_u \geq \frac{\hbar}{2\Delta u}, \quad \Delta p_u \geq \frac{\hbar a}{2}\Delta u. \qquad (16)$$

The uncertainty relations (16) can be combined linearly to give

$$\Delta p_u \geq \frac{\hbar}{2}\left(\frac{1}{\Delta u} + Da\Delta u\right), \tag{17}$$

where D denotes an undefined constant. This relation entails a minimal uncertainty for p_u [6]

$$(\Delta p_u)_{min} = \hbar\sqrt{Da}. \tag{18}$$

The uncertainty relations for the transverse components of the momentum operator are more complex, those for Δp_v remain unmodified.

5 Some remarks about the quantum field theory

We have presented possible kinematics of a relativistic quantum test particle in a curved space-time. In a flat space-time, the full first-quantized theory of a relativistic particle is plagued by negative probabilities and must be replaced by a second-quantized quantum field theory.

We may try to mimic the flat space-time quantum field theory by replacing the momentum operators $\mathbf{P}_k = -i\hbar\partial_k$ with the generalized momentum operators defined via the infinitesimal right geodesic translation operators $\mathbf{p}_k = -i\hbar R^s_k(x)\partial_s$. In a flat space-time, components of the momentum operator commute, $[\mathbf{P}_j, \mathbf{P}_k] = 0$. In a curved background we have instead,

$$[\mathbf{p}_j, \mathbf{p}_k] = -i\hbar\rho^n_{jk}(x)\,\mathbf{p}_n. \tag{19}$$

For instance, in the case of a weak plane gravitational wave background, in the approximations considered in the last section we have two nonvanishing commutators:

$$\begin{aligned}
[\mathbf{p}_y, \mathbf{p}_u] &= \frac{i\hbar}{2}A\omega(2\mathbf{y}\mathbf{p}_v - \mathbf{u}\mathbf{p}_y), \\
[\mathbf{p}_z, \mathbf{p}_u] &= -\frac{i\hbar}{2}A\omega(2\mathbf{z}\mathbf{p}_v + \mathbf{u}\mathbf{p}_z).
\end{aligned}$$

It follows that the time-like component of the momentum $\mathbf{p}_t = \mathbf{p}_u + \mathbf{p}_v$ doesn't commute with the transverse components of the momentum, $[\mathbf{p}_t, \mathbf{p}_y] \neq 0$, $[\mathbf{p}_t, \mathbf{p}_z] \neq 0$. If $\mathbf{p}_t$ could be interpreted as the energy operator (hamiltonian) and if an analog of the Noether theorem for geodesic loops could be established, this means that the transverse momentum of the quantum field is not conserved in the background of a weak plane gravitational wave. There has been some progress in establishing generalized conservation laws in the case of Moufang loops [12], but nothing can be said in the case of more general geodesic loops. The representation theory of general nonassociative structures is also essentially lacking and we cannot introduce one- and many-particle states as suitable representations of geodesic loops. It seems that some novel mathematical ideas and developments are needed for continuing our investigations in this direction.

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